Generalized Gravitational S-Duality and the Cosmological Constant Problem

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Abstract

We study S-duality transformations that mix the Riemann tensor with the field strength of a 3-form field. The dual of an $(A)dS$ space time – with arbitrary curvature – is seen to be flat Minkowski space time, if the 3-form field has vanishing field strength before the duality transformation. It is discussed whether matter could couple to the dual metric, related to the Riemann tensor after a duality transformation. This possibility is supported by the facts that the Schwarzschild metric can be obtained as a suitable contraction of the dual of a Taub-NUT-AdS metric, and that metrics describing FRW cosmologies can be obtained as duals of theories with matter in the form of torsion.

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1 Introduction

The cosmological constant problem has motivated various attempts to modify Einstein's theory of gravity. In the present paper we study S-duality acting on the gravitational field. The presence of a dual graviton is motivated by hidden symmetries in $d = 11$ supergravity/M-theory [1].

Subsequently we confine ourselves to $d = 4$ dimensions, where the dual of a graviton is again a graviton-like symmetric two-component tensor field [2–6]. We want to persue the question whether a metric obtained through such a duality transformation can describe naturally a space time which is flat (rather than (A)dS), although the original space time (before the duality transformation) is (A)dS with arbitrary cosmological constant. The idea is that, although space-time is possibly strongly (A)dS in one version of gravity, we "see" its dual that is obtained by the above duality transformation.

In order to describe the concept behind this approach it is best to start with the well understood concept of duality in electromagnetism. If A is an abelian (one form) gauge field, F its (two form) field strength, the duality transformation relating A and F to its duals $\widetilde{A}$ and $\widetilde{F}$ reads

$$\widetilde{F}(\widetilde{A}) = \star F(A) \; . \tag{1.1}$$

Whereas a tensor $\widetilde{F}$ can always be defined through eq. (1.1), it can be expressed in terms of a dual gauge field $\widetilde{A}$ only if $d\widetilde{F} = 0$, i.e. if $F(A)$ satisfies the (free) equations of motion $d \star F = 0$.

The introduction of the dual gauge fields $\widetilde{A}$ would be particularly useful, if magnetic monopoles would exist: It is not possible to couple a magnetic monopole (described by a magnetic current J_M) locally to A. On the other hand, such a magnetic current couples locally to $\widetilde{A}$ like $\widetilde{A} \cdot J_M$ in the same way as an electric current couples to A. If only magnetic monopoles would exist, but no electrically charged particles, it would be reasonable to replace A by $\widetilde{A}$ everywhere in the theory. Since electromagnetism is self dual, the resulting theory (written in terms of $\widetilde{A}$) would look the same as a theory with electrically charged particles written in terms of A, and one would simply denote the original magnetic monopoles as

electric monopoles.

Already in the relatively simple case of electromagnetism it is not quite trivial to find a parent action $S(A, \widetilde{A})$ that allows to obtain the duality relation (1.1) as a consequence of the equations of motion: Such a parent action is either not manifestly Lorentz covariant [7, 8] or includes additional auxiliary fields [9, 10].

On the other hand, a parent action is not necessary in order to define a duality transformation relating A and $\widetilde{A}$: It is sufficient to define such a relation via (1.1), and to verify that equations of motion and Bianchi identities for F and $\widetilde{F}$ get interchanged.

In the present paper we persue the second approach in order to define dual gravity. Subsequently it will be useful to work with tensors with (latin) indices in (flat) tangent space that are related to tensors with (greek) space time indices, as usual, by contractions with a vierbein. A duality relation analogous to (1.1) reads then [2–4, 6]

$$\widetilde{R}_{abcd}(\widetilde{e}) = \frac{1}{2}\, \varepsilon_{abef}\ R^{ef}{}_{cd}(e) + \ldots \tag{1.2}$$

where the Riemann tensor R_{abcd} depends on a vierbein e (or a metric g) as usual, and the dual Riemann tensor $\widetilde{R}_{abcd}$ is assumed to depend similarly on a dual vierbein $\widetilde{e}$ (or a dual metric $\widetilde{g}$) (the dots in (1.2) are introduced for later porposes).

The standard coupling of gravity to matter is of the form

$$G_{ab} = -T_{ab} \tag{1.3}$$

where G_{ab} is the Einstein tensor, and T_{ab} the stress energy tensor of matter. For most practical purposes (tests of general relativity) it suffices to consider matter in the form of point like particles. Either such matter serves as a "source" for the gravitational field (as the Schwarzschild solution), or as test particles: From the covariant conservation of the stress energy tensor T_{ab} (which has its origin in the Bianchi identity for the Riemann tensor R_{abcd}) it follows that any point like object, that couples like (1.3) to gravity, moves along geodesics corresponding to the metric g.

Also stress energy tensors of the Friedmann-Robertson-Walker form, that involve a matter density $\rho(t)$ and a pressure $p(t)$, can be understood as suitable averages over pointlike sources

[11].

The cosmological constant problem arises from contributions of the form $\sim\ \eta_{ab}\Lambda$ to T_{ab}, which have its origin in the minimal coupling of fields (the fields of the standard model) to gravity: only then potentials of classical scalar fields (Higgs fields), vacuum condensates (as in QCD) and quantum contributions to the expectation value of T_{ab} generate values of Λ that gives rise, via (1.3), to an unobserved space time curvature $\sim \Lambda$. We are very far, however, from being able to test the minimal coupling of the fields of the standard model to gravity.

Now, if dual gravity in the sense of eq. (1.2) exists, it would be interesting to investigate whether macroscopic matter, that is involved in classical tests of general relativity, may couple to a dual metric $\widetilde{g}$ in the form

$$\widetilde{G}_{ab} = -\widetilde{T}_{ab}\ . \tag{1.4}$$

Then, since $\widetilde{T}_{ab}$ is covariantly conserved with the dual metric in the covariant derivative, matter would move on geodesics corresponding to the dual metric $\widetilde{g}$, and we would "see" a space time metric $\widetilde{g}$.

Clearly, here we have to assume that the duality transformation (1.2) is accompagnied by a duality transformation acting on the stress energy tensor such that the dual stress energy tensor is consistent with (1.4). We can not expect, on the other hand, that duality transformations including gravity – beyond the linearized level – can be derived from a local parent action that would allow to derive the contributions of matter fields to the dual stress energy tensor from a variational principle. We will thus proceed with the assumption that a more complete (probably non local) consistent duality transformation – leading to a stress energy tensor consistent with (1.4) – exists and use, for the time being, the results of the duality transformations (1.2) to define the action of duality on the stress energy tensor. (We return to this issue in sections 3 and, in the particular case $\widetilde{T}_{ab} = 0$, in section 6.)

In the following we want to investigate, whether eq. (1.4) contradicts eq. (1.2) (leaving aside a possible microscopic origin of eq. (1.4)): At first sight it seems that we just have to rename $\widetilde{g}$ to g in order to recover standard Einstein gravity from eq. (1.4). However, if we require simultaneously the validity of eq. (1.2), it is not clear whether this does not restrict

the possible configurations of $\widetilde{g}$ to an inacceptable level.

It is already known that, in the vacuum (where $T_{ab} = \widetilde{T}_{ab} = 0$) and in the weak field limit, this is not the case [2, 3]: in this limit gravity is self dual, i.e. for every Riemann tensor on the right hand side of eq. (1.2) that solves the vacuum equations of motion (whose Ricci tensor vanishes), one can define a dual Riemann tensor, derivable from a dual metric, that solves the vacuum equations of motion as well. (For a corresponding parent actions quadratic in the fields see, e.g., refs. [4–6] and, in the case of the Macdowell-Mansouri formalism, ref. [2]. In the latter case, where a cosmological constant is present, it is argued that the duality transformation leads to an inversion of the cosmological constant.)

As in the case of Yang-Mills theories the real problems arise, however, at the nonlinear level: At the nonlinear level we do not know how to map $g \to \widetilde{g}$ such that the corresponding Riemann tensors satisfy eq. (1.2), and such that equations of motion and Bianchi identities get interchanged. On the other hand, only at the nonlinear level the covariant derivatives (with respect to which the Bianchi identities hold, and hence with respect to which the corresponding Einstein tensors and hence the corresponding stress energy tensors are conserved) involve the connections of the corresponding metrics g or $\widetilde{g}$, and imply thus the motion of matter on corresponding geodesics.

Attempts to proceed via parent actions beyond the linearized level run into conflicts with no-go theorems on interacting theories (with at most two derivatives) of "dual" gravitons [12]. Therefore we will confine ourselves to duality relations at the level of equations of motion (and Bianchi identities) of the form of eq. (1.2) allowing eventually for additional contributions on the right hand side. We will investigate whether particular configurations of a dual metric $\widetilde{g}$, that are of confirmed phenomenological relevance (the Schwarzschild metric, and Freedman-Robertson-Walker (FRW) like cosmologies), are consistent with a suitably generalized (see below) duality transformation at the nonlinear level.

In general, once several gauge fields (and hence corresponding field strengths) are present in a theory, these can mix under duality transformations [13, 1] if their rank is appropriate. Examples are gauge fields of higher rank in d=10 and d=11 supergravity theories [13, 1]. In [1] it has been proposed that the corresponding algebras include gravity in a nontrivial way. It seems then possible that also the field strength of gravity (the Riemann tensor) mixes

with other field strengths under a duality transformation. Indeed, a three form gauge field (with a four form field strength) is present in these higher dimensional supergravity theories [13, 1], which has the appropriate rank.

Below we show that a modification of the duality transformation (1.2), that mixes the Riemann tensor with a four form field strength, is consistent at the same level (linear in the gravitational excitations) as duality in pure gravity, in the sense that it exchanges equations of motion and Bianchi identities.

Moreover, this modified duality transformation is seen to have several virtues: Notably, it transforms the Riemann tensor of a space time metric g with arbitrary curvature ((A)dS space times) into a Riemann tensor of a space time $\widetilde{g}$ whose curvature is given by the vev of the four form field strength before the duality transformation. If this vev vanishes, the dual space time described by $\widetilde{g}$ is thus flat (Minkowski), irrespectively of the curvature of the original space time described by g. This may point towards an unconventional solution of the cosmological constant problem, if macroscopic matter couples to dual gravity as discussed above.

This mechanism to obtain a vanishing cosmological constant is very different from its cancellation by a specific (fine tuned) value for the vev of the four form field strength as considered in [14], and also from the proposal in [15]: Here we suggest that, although space-time is possibly strongly (A)dS in one version of gravity, we "see" its dual that is obtained by the above duality transformation.

Next, it is impossible to obtain a Schwarzschild metric $\widetilde{g}$ as the dual of a metric g (that solves the vacuum equations of motion) using the "standard" gravitational duality transformation. At the linearized level, a Schwarzschild metric $\widetilde{g}$ with mass $\widetilde{m}$ is dual to a Taub-NUT metric g with NUT parameter $\ell = \widetilde{m}$ (and mass $m = 0$) [3, 4]. However, this simple relation does not continue to hold at the full nonlinear level, since a metric $\widetilde{g}(\widetilde{m}, \widetilde{\ell})$ that is dual to a Taub-NUT metric $g(m, l)$ (in the sense of eq. (1.2)) becomes singular in the limit $\widetilde{\ell} \to 0$ or $m \to 0$ [16]. Using the modified duality transformation rule for the Riemann tensor, we find an exact relation between Taub-NUT-(A)dS metrics $\widetilde{g}(\widetilde{m}, \widetilde{\ell}, \widetilde{\Lambda})$ and $g(m, \ell, \Lambda)$. Only then we find that a Schwarzschild metric $\widetilde{g}(\widetilde{m}, \widetilde{\ell} \to 0, \widetilde{\Lambda} \to 0)$ in Minkowski space can be obtained as the dual of a suitable contraction of a Taub-NUT-(A)dS metric

$g(m \to 0, \ell \to 0, \Lambda \to -\infty)$ (with m/ℓ and $m^3\Lambda$ fixed).

However, the modified duality transformation rule for the Riemann tensor including a four form field strength is still not sufficient to allow for dual metrics $\widetilde{g}$ of the form of FRW cosmologies. The reason is that for such metrics $\widetilde{g}$ the Ricci tensor is not constant, but the above duality transformation relates the Ricci tensor of $\widetilde{g}$ to the first Bianchi identity (the cyclic identity), up to a constant, of the original Riemann tensor. A solution of this problem consists in a further modification of the duality transformation: In the space time described by g, we propose to couple matter to gravity in the form of torsion in the Riemann tensor $R_{abcd}(g)$.

A priori, in the presence of torsion in $R_{abcd}(g)$ it is still highly non-trivial to generate a Riemann tensor $\widetilde{R}_{abcd}(\widetilde{g})$ through a duality transformation, that is torsion free and can be derived from a FRW like metric $\widetilde{g}$. Nevertheless it turns out that a quite simple ansatz for the torsion in $R_{abcd}(g)$ – represented as non-metric contributions to the connection – does the job: it suffices to include torsion in the form of a vector and an axial vector, whose only nonvanishing components are its time like components $\gamma(t)$ and $\beta(t)$, respectively. Including in addition a FRW like metric g with a scale factor $a(t)$, one finds that two relations among $a(t)$, $\gamma(t)$ and $\beta(t)$ are sufficient in order to generate a dual Riemann tensor that can be derived from a FRW metric $\widetilde{g}$ with an arbitrary scale factor $\widetilde{a}(t)$. The non-standard form of the duality relation plays a crucial role to this end.

The subsequent outline of the paper is as follows: In the next section 2 we review the properties of standard (linearized) gravitational S-duality. In section 3 we present a non-standard gravitational duality rule including a 3-form field. We discuss its consistency at the linearized level, and derive the result mentioned above: under a simple assumption flat Minkowski space appears as the dual of (A)dS, for any value of the de Sitter curvature.

In section 4 we generalize this result to Taub-NUT-(A)dS metrics, and derive the Schwarzschild metric as a contraction of a dual Taub-NUT-AdS metric.

In section 5 we consider Riemann tensors with torsion, and derive FRW cosmologies as duals of theories with torsion. In section 6 we conclude with a discussion.

2 The Dual of Gravity

For most of the paper it will be convenient to work with tensors with (latin) indices in (flat) tangent space, that are related to tensors with (greek) indices, as usual, by contractions with a vierbein. For the Riemann tensor this relation reads

$$R_{abcd} = e_a^{\ \mu} \ e_b^{\ \nu} \ e_c^{\ \rho} \ e_d^{\ \sigma} \ R_{\mu\nu\rho\sigma} \ . \tag{2.1}$$

Let us recall the symmetry properties of R_{abcd}:

$$R_{abcd} = -R_{bacd} = -R_{abdc} = +R_{cdab} \ . \tag{2.2}$$

It satisfies the first Bianchi identity (or cyclic identity)

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{2.3}$$

and the second Bianchi identity

$$D_e R_{abcd} + D_c R_{abde} + D_d R_{abec} = 0 \ . \tag{2.4}$$

In the vacuum, the equations of motion imply the vanishing of the Ricci tensor:

$$R^a_{\ b} \equiv R^{ca}_{\ \ bc} = 0 \tag{2.5}$$

where indices are raised and lowered with the flat metric $\eta_{ab} = diag(-1,\ 1,\ 1,\ 1)$.

A dual Riemann tensor $\widetilde{R}_{abcd}$ is obtained from R_{abcd} by a contraction with the antisymmetric tensor ε_{abcd}:

$$\widetilde{R}_{abcd} = \frac{1}{2} \ \varepsilon_{abef} \ R^{ef}_{\ \ cd} \qquad \text{or} \qquad \widetilde{R}_{abcd} = \frac{1}{2} \ R_{ab}^{\ \ ef} \ \varepsilon_{efcd} \ . \tag{2.6}$$

The properties of $\widetilde{R}_{abcd}$ have previously been discussed in [3–6]. Its first Bianchi identity follows from the vanishing of the Ricci tensor (2.5) corresponding to R_{abcd}. Its second Bianchi identity *at the linearized level* can be derived from the second Bianchi identity of R_{abcd} (*at the linearized level*) if, again, $R^a_{\ b}$ vanishes. Finally the first Bianchi identity for R_{abcd}, eq. (2.3), implies the vanishing of the dual Ricci tensor.

Its symmetries together with the Bianchi identities are sufficient to prove that, at the linearized level, $\widetilde{R}^{lin}_{\mu\nu\rho\sigma}$ can be written in terms of a dual linearized metric $\widetilde{h}_{\mu\nu}$ [18] (the distinction between latin and greek indices is meaningless at the linearized level) as

$$\widetilde{R}^{lin}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\widetilde{h}_{\mu\sigma,\nu\rho} + \widetilde{h}_{\nu\rho,\mu\sigma} - \widetilde{h}_{\mu\rho,\nu\sigma} - \widetilde{h}_{\nu\sigma,\mu\rho}\right) \ . \tag{2.7}$$

An explicit formula for $\widetilde{h}_{\mu\nu}$ in terms of $\widetilde{R}_{\mu\nu\rho\sigma}$ is given in [18] in the coordinate gauge $x^\mu \widetilde{h}_{\mu\nu} = x^\nu \widetilde{h}_{\mu\nu} = 0$:

$$\widetilde{h}_{\mu\nu}(x) = -\int_0^1 dt \int_0^t dt' t' \ x^\rho x^\sigma \ \widetilde{R}_{\mu\rho\nu\sigma}(t'x) \ . \tag{2.8}$$

Thus the S-dual of linearized gravity can be constructed explicitly. The validity of the second Bianchi identity (2.4) for the dual Riemann tensor beyond the linearized level requires, however, the knowledge of the dual connections which are not yet constructed at this point, and this problem has no general solution.

3 The S-Dual of Gravity and a 3-Form Field

The standard gravitational S-duality transformation (2.6) allows only to relate metrics with vanishing Ricci tensors [2, 3]. In the present section we present a modified duality transformation that allows to relate metrics whose Ricci tensors $R^a{}_b$ satisfy

$$R^b{}_a \equiv R^{ca}{}_{bc} = \Lambda \delta^a{}_b \ . \tag{3.1}$$

As stated in the introduction, we assume the presence of a three form field $A_{abc} = A_{[abc]}$, with field strength $F_{abcd} = \partial_{[a} A_{bcd]}$ and equation of motion $\partial^a F_{abcd} = 0$, and study a modified duality transformation that mixes the Riemann tensor with F_{abcd}. As general solution of the equation of motion of the three form field we can take

$$F_{abcd} = \Sigma \varepsilon_{abcd} \qquad , \qquad \Sigma = \text{const.} \ . \tag{3.2}$$

The proposed generalization of the duality transformation (2.6) reads:

$$\widetilde{R}_{abcd} = \frac{1}{2}\varepsilon_{abef}\left(R^{ef}{}_{cd} + F^{ef}{}_{cd}\right) + \frac{1}{12}\ \varepsilon_{abcd} R \ , \tag{3.3a}$$

$$\widetilde{F}_{abcd} = -\frac{1}{12}\ \varepsilon_{abcd} R \tag{3.3b}$$

(or with the reversed order of ε and $(R+F)$ in (3.3a), cf. the second of eqs. (2.6)), where $R \equiv R^{ab}{}_{ba}$.

In order to justify eqs. (3.3) we have to show that they imply an exchange of equations of motion with Bianchi identities, and that a double duality transformation acts as minus the identity (in space time with Minkowski signature).

Let us first discuss the Bianchi identities to be satisfied by the dual Riemann tensor $\widetilde{R}_{abcd}$ in (3.3a) (the Bianchi identity for $\widetilde{F}_{abcd}$ is trivial in d=4). Using (3.2) and the gravitational equation (3.1) it is straightforeward to show that $\widetilde{R}_{abcd}$ has the symmetry properties (2.2), and satisfies the first Bianchi identity (2.3). As in the case of the "standard" duality relation (2.6), the validity of the second Bianchi identity (2.4) can only been shown at the linearized level, where one has to use the linearized second Bianchi identity for R_{abcd}, and the fact that both the Ricci tensor $R^a{}_b$ and F_{abcd} are constant.

These properties of $\widetilde{R}_{abcd}$ are sufficient to prove that, at the linearized level, it can again be expressed in terms of a dual metric $\widetilde{h}_{\mu\nu}$ as in eq. (2.7).

Now we turn to the equations of motion satisfied by the dual tensors. For the dual Ricci tensor one obtains

$$\widetilde{R}^a{}_b = 3\Sigma\delta^a{}_b \equiv \widetilde{\Lambda}\delta^a{}_b \tag{3.4}$$

with the help of the first Bianchi identity for R_{abcd}, and eq. (3.2) for F_{abcd}.

For the dual four form field strength $\widetilde{F}_{abcd}$ one finds, from $R = 4\Lambda$ and eq. (3.3b),

$$\widetilde{F}_{abcd} = -\frac{1}{3}\Lambda\varepsilon_{abcd} \equiv \widetilde{\Sigma}\varepsilon_{abcd}\ , \tag{3.5}$$

which is indeed a solution of the dual equations of motion for $\widetilde{A}_{abc}$.

Equations (3.4) and (3.5) show that in some sense A_{abc} is dual to the cosmological constant: Up to a factor 3 the duality transformations (3.3) lead to an interchange of Σ, the parameter characterizing the solution of the equation of motion of A_{abc}, with the cosmological constant Λ.

As stated in the introduction, here we have to assume that the duality transformations (3.3) are accompagnied by duality transformations acting on the stress energy tensor such

that the dual stress energy tensor is consistent with $\widetilde{\Lambda} = 3\Sigma$. Note that, in the absence of a parent action, we cannot determine a priori the contribution of F_{abcd} to the stress energy tensor (or its dual) – this contribution is not fixed by the equation of motion $\partial^a F_{abcd} = 0$, but implicitely by the gravitational equation of motion (3.4).

Finally we have to check whether a double duality transformation reproduces (minus) the identity. After some calculation one obtains indeed

$$\widetilde{\widetilde{R}}_{abcd} = -R_{abcd} \tag{3.6}$$

and

$$\widetilde{\widetilde{F}}_{abcd} = -F_{abcd} \ . \tag{3.7}$$

Hence the duality transformations (3.3) have all the desirable properties. As before, however, the validity of a second Bianchi identity for $\widetilde{R}_{abcd}$ can not be proven beyond the linearized level.

Let us make a comment on the need to include the three form field A_{abc} in the duality relation (3.3a). At first sight, we could have omitted F_{abcd} in (3.3a), and dropped (3.3b): Then $\widetilde{R}_{abcd}$ would still satisfy the Bianchi identities (at the same linearized level as before), and the gravitational equations of motion with $\widetilde{R}^a{}_b = 0$. However, then a double duality transformation would reproduce (minus) the identity on R_{abcd} only iff $\Lambda = 0$. It is the validity of (3.6) for Λ arbitrary that forces us to include F_{abcd} in (3.3a).

Now we make the following evident, but important, observation: *Iff* the vev Σ of the 3-form field strength (before the duality transformation) vanishes, the dual Ricci tensor vanishes by virtue of eq. (3.4), independently from the value Λ of the space-time described by the Riemann tensor R_{abcd} before the duality transformation. Hence, *iff* for some reason we "see" the space-time described by the dual Riemann tensor $\widetilde{R}_{abcd}$, we then see a space-time with vanishing cosmological constant.

As discussed in the introduction, we may identify $\widetilde{R}_{abcd}$ with our "physical" space-time only if $\widetilde{R}_{abcd}$ can also describe physically relevant non-trivial configurations as the Schwarzschild and FRW metrics, and this beyond the linearized level (such that the full second Bianchi identity (2.4) holds). The analysis of the action of the generalized duality transformations (3.3a) on metrics that are suitable generalizations of the Schwarzschild

metric is the subject of the next chapter.

4 Non-standard Duality and Taub-NUT-(A)dS Spaces

At the level of linearized standard gravitational S-duality, the parameters m and ℓ of a Taub-NUT metric [19, 20] get interchanged [3, 4]. Hence the NUT parameter ℓ can be interpreted as a "magnetic" mass. On Taub-NUT spaces the gravitational S-duality can be extended to the full nonlinear level [16], and this can be generalized to Taub-NUT-(A)dS spaces in the case of the non-standard gravitational S-duality (3.3) [16]. At the nonlinear level, however, the relations between the original parameters m, ℓ, and the parameters $\widetilde{m}$, $\widetilde{\ell}$ caracterizing the dual configuration, are somewhat more involved (see below).

The Taub-NUT-(A)dS metric can be written as the following generalization of the Taub-NUT metric [17]:

$$ds^2 = -f^2(r)\left(dt + 4\ell\sin^2\frac{\theta}{2}d\phi\right)^2 + f^{-2}(r)dr^2 + \left(r^2+\ell^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.1}$$

with

$$f^2(r) = 1 - \frac{2\left(mr+\ell^2\right) - \Lambda\left(\frac{1}{3}r^4 + 2\ell^2r^2 - \ell^4\right)}{r^2+\ell^2}\ . \tag{4.2}$$

The non-vanishing components of the Riemann tensor are

$$\begin{aligned}
R_{0101} &= -2\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) + \frac{1}{3}\Lambda\\
R_{0202} = R_{0303} &= \left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) + \frac{1}{3}\Lambda\\
R_{1212} = R_{1313} &= -\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) - \frac{1}{3}\Lambda\\
R_{2323} &= 2\left(1+\frac{4}{3}\Lambda\ell^2\right)A_{\bar{m},\ell}(r) - \frac{1}{3}\Lambda\\
R_{0312} = -R_{0213} &= \left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r)\\
R_{0123} &= -2\left(1+\frac{4}{3}\Lambda\ell^2\right)D_{\bar{m},\ell}(r)\ ,
\end{aligned} \tag{4.3}$$

where $A_{\bar{m},\ell}$ and $D_{\bar{m},\ell}$ are given by

$$A_{\bar{m},\ell}(r) = \frac{\bar{m}r^3 + 3\ell^2r^2 - 3\bar{m}\ell^2r - \ell^4}{(r^2+\ell^2)^3}\ ,$$

$$D_{\bar{m},\ell}(r) = \frac{-\ell r^3 + 3\ell\bar{m}r^2 + 3r\ell^3 - \bar{m}\ell^3}{(r^2+\ell^2)^3} \tag{4.4}$$

and

$$\bar{m} = m\left(1 + \frac{4}{3}\Lambda\ell^2\right)^{-1} \ . \tag{4.5}$$

Constructing the components of the dual Riemann tensor from eq. (3.3a) one obtains contributions from the terms $\sim F_{abcd}$ and $\sim R$. One finds

$$\begin{aligned}
\widetilde{R}_{0101} &= -2\left(1 + \frac{4}{3}\Lambda\ell^2\right) D_{\bar{m},\ell}(r) + \Sigma \\
\widetilde{R}_{0202} = \widetilde{R}_{0303} &= \left(1 + \frac{4}{3}\Lambda\ell^2\right) D_{\bar{m},\ell}(r) + \Sigma \\
\widetilde{R}_{1212} = \widetilde{R}_{1313} &= -\left(1 + \frac{4}{3}\Lambda\ell^2\right) D_{\bar{m},\ell}(r) - \Sigma \\
\widetilde{R}_{2323} &= 2\left(1 + \frac{4}{3}\Lambda\ell^2\right) D_{\bar{m},\ell}(r) - \Sigma \\
\widetilde{R}_{0312} = \widetilde{R}_{0213} &= -\left(1 + \frac{4}{3}\Lambda\ell^2\right) A_{\bar{m},\ell}(r) \\
\widetilde{R}_{0123} &= 2\left(1 + \frac{4}{3}\Lambda\ell^2\right) A_{\bar{m},\ell}(r) \ .
\end{aligned} \tag{4.6}$$

The following properties of the functions $A_{m,\ell}$, $D_{m,\ell}$ are helpful in order to find a metric that reproduces the Riemann tensor (4.6): If one defines

$$m' = -\frac{\ell^2}{m} \tag{4.7}$$

one has

$$\begin{aligned}
A_{m',\ell}(r) &= \ \frac{\ell}{m}\ D_{m,\ell}(r) \ , \\
D_{m',\ell}(r) &= -\frac{\ell}{m}\ A_{m,\ell}(r) \ .
\end{aligned} \tag{4.8}$$

Note that, at the level of linearized gravity, we could replace $A_{m,\ell}(r)$ and $D_{m,\ell}(r)$ in (4.4) by their asymptotic forms for $r \to \infty$. Then we would obtain the simple relation $A_{m,\ell}(r) = -D_{\ell,m}(r)$. This simple relation does not survive in full non-linear gravity.

Using the relations (4.8) and the definition (4.5) of $\bar{m}$ one finds that the following metric reproduces all components of $\widetilde{R}_{abcd}$:

$$\widetilde{ds}^2 = -\frac{\ell}{m - 4\Sigma\ell^3}\left\{\widehat{f}^2(r)\left(dt + 4\ell\sin^2\frac{\theta}{2}d\phi\right)^2 - \left[\widehat{f}^{-2}(r)dr^2 + \left(r^2+\ell^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]\right\} \tag{4.9}$$

with

$$\widehat{f}^{\,2}(r) = 1 + \frac{-2\ell^2\left(m - 4\Sigma\ell^3 - r\left(1 + \frac{4}{3}\Lambda\ell^2\right)\right) + 3\Sigma\ell\left(\frac{1}{3}r^4 + 2r^2\ell^2 - \ell^4\right)}{\left(m - 4\Sigma\ell^3\right)\left(r^2 + \ell^2\right)} \ . \tag{4.10}$$

In order to bring this metric into the same form as in (4.1) one has to rescale the coordinates as

$$t = \sqrt{\frac{m - 4\Sigma\ell^3}{\ell}}\; t' \qquad , \qquad r = \sqrt{\frac{m - 4\Sigma\ell^3}{\ell}}\; r' \ , \tag{4.11}$$

and to define the dual parameters

$$\begin{aligned} &\widetilde{m} = -\left(1 + \frac{4}{3}\Lambda\ell^2\right)\left(m - 4\Sigma\ell^3\right)^{-3/2}\ell^{5/2} \quad , \quad \widetilde{\ell} = \ell^{3/2}\left(m - 4\Sigma\ell^2\right)^{-\frac{1}{2}} \ , \\ &\widetilde{\Lambda} = 3\Sigma \ . \end{aligned} \tag{4.12}$$

This allows to write the dual metric again (up to an overall sign) in the form (4.1) with

$$\widetilde{f}^2(r') = 1 - \frac{2\left(\widetilde{m}r' + \widetilde{\ell}^2\right) - \widetilde{\Lambda}\left(\frac{1}{3}r'^4 + 2r'^2\widetilde{\ell}^2 - \widetilde{\ell}^4\right)}{r'^2 + \widetilde{\ell}^2} \ . \tag{4.13}$$

Thus the metric dual to a Taub-NUT-(A)dS metric is again of the Taub-NUT-(A)dS form. Let us now assume that the vev Σ of the 3-form field strength vanishes. Then the expressions (4.12) for the dual parameters collapse to

$$\widetilde{m} = -\left(1 + \frac{4}{3}\Lambda\ell^2\right) m^{-3/2}\ell^{5/2} \ , \ \widetilde{\ell} = \ell^{3/2}m^{-\frac{1}{2}} \ , \ \widetilde{\Lambda} = 0 \ . \tag{4.14}$$

Now, as in section 3, the dual cosmological constant vanishes, but, somewhat disturbingly, the dual NUT parameter $\widetilde{\ell}$ does not vanish for $m \to 0$ (in contrast to its behaviour in linearized gravity). However, a vanishing dual NUT parameter – as required for a dual Schwarzschild metric – can be obtained in the following limit:

$$\Lambda \to -\infty, \quad m,\ \ell \to 0, \quad m/\ell = k = const. \tag{4.15}$$

It turns out that the constant k can be absorbed into a rescaling of the coordinates and be chosen as $k = 1$. Then, in addition, we require

$$-\frac{4}{3}m^3\Lambda = \widetilde{m} = const. \tag{4.16}$$

when taking the limits (4.15). Now, since $\widetilde{\ell} \to 0$, the dual metric (as described by $\widetilde{f}^2(r')$ in (4.13) with $\widetilde{\ell} = \widetilde{\Lambda} = 0$) coincides with the Schwarzschild metric with mass $\widetilde{m}$.

Note that during the above contraction of the original metric we have kept the coordinates r, t constant, which is a coordinate dependent statement. As usual in the case of contractions, coordinates have eventually be rescaled after a parameter dependent general coordinate transfromation.

Hence we have obtained the desired result: the Schwarzschild metric can be obtained as the dual of a contracted Taub-NUT-AdS metric. This result would not have been possible in the absence of an "original" cosmological constant Λ, and using the standard gravitational S-duality transformation. Note that although the original metric (and the components of the original Riemann tensor) diverge in the above limit (4.15), these infinities cancel in the non-standard expression (3.3a) for the dual Riemann tensor which is what makes this result possible.

5 FRW Cosmologies as Duals of Gravity with Torsion

In this section we investigate whether a dual Riemann tensor $\widetilde{R}_{abcd}$, obtained through a non-standard duality transformation of the type (3.3a), can be identified with a Riemann tensor describing FRW cosmologies. FRW cosmologies correspond to a metric

$$d\widetilde{s}^2 = -dt^2 + \widetilde{a}^2(t)\ d\vec{x}^2 \tag{5.1}$$

where, of course, $\widetilde{a}(t)$ depends on the properties of the matter to which the Einstein tensor couples.

Defining

$$\widetilde{a}(t) = e^{\widetilde{\alpha}(t)} \tag{5.2}$$

the only nonvanishing components of the Riemann tensor $\widetilde{R}_{abcd}$ are

$$\widetilde{R}_{ijij} = \dot{\widetilde{\alpha}}^2 \text{ (no sum over } i,\ j) \ , \tag{5.3a}$$

$$\widetilde{R}_{i0i0} = -\dot{\tilde{\alpha}}^2 - \ddot{\tilde{\alpha}} \ , \tag{5.3b}$$

and the nonvanishing components of the Ricci tensor are

$$\widetilde{R}_{ii} = -3\dot{\tilde{\alpha}}^2 - \ddot{\tilde{\alpha}} \ , \tag{5.4a}$$

$$\widetilde{R}_{00} = 3\dot{\tilde{\alpha}}^2 + 3\ddot{\tilde{\alpha}} \ , \tag{5.4b}$$

where dots denote time derivatives.

However, the duality transformation (3.3a) allows only for Ricci tensors $\widetilde{R}_{ab} = 3\Sigma\eta_{ab}$ (see (3.4)) with $\dot{\Sigma} = 0$ from the equation of motion (3.2) for the 3-form field. Hence the duality transformation (3.3a) has to be modified by additional terms corresponding to contributions from matter in the "original" version of the theory before the duality transformation.

The most elegant way to do this is to replace the Riemann tensor R_{abcd} on the right-hand side of (3.3a) by a Riemann tensor including torsion [21]. Our corresponding conventions are as follows: the Riemann tensor is written as

$$R^{\sigma}{}_{\mu\nu\rho} = \Gamma^{\sigma}{}_{\mu\rho,\nu} - \Gamma^{\sigma}{}_{\mu\nu,\rho} + \Gamma^{\sigma}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\rho} - \Gamma^{\sigma}{}_{\beta\rho}\Gamma^{\beta}{}_{\mu\nu} \tag{5.5}$$

where the connection is decomposed as

$$\Gamma^{\sigma}{}_{\mu\nu} = {}^{M}\Gamma^{\sigma}{}_{\mu\nu} + \widehat{\Gamma}^{\sigma}{}_{\mu\nu} \ . \tag{5.6}$$

Here ${}^{M}\Gamma^{\sigma}{}_{\mu\nu}$ is the standard connection constructed from the metric $g_{\mu\nu}$, and $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$ represents torsion. Requiring $g_{\mu\nu;\rho} = 0$ (where the covariant derivative is defined with the full connection $\Gamma^{\sigma}{}_{\mu\nu}$) implies

$$\widehat{\Gamma}_{\sigma\mu\nu} = \widehat{\Gamma}_{[\sigma\mu]\nu} \ , \tag{5.7}$$

where indices are raised and lowered with the metric $g_{\mu\nu}$. Assuming eq. (5.7), $\widehat{\Gamma}_{\sigma\mu\nu}$ can be decomposed with respect to the Lorentz group as [21–23]

$$\widehat{\Gamma}_{\sigma\mu\nu} = \widehat{\Gamma}^{V}_{\sigma\mu\nu} + \widehat{\Gamma}^{A}_{\sigma\mu\nu} + \widehat{\Gamma}^{T}_{\sigma\mu\nu} \ . \tag{5.8}$$

Here $\widehat{\Gamma}^V_{\sigma\mu\nu}$ is proportional to a vector V_μ,

$$\widehat{\Gamma}^V_{\sigma\mu\nu} = V_{[\sigma}\ g_{\mu]\nu}\ , \tag{5.9}$$

$\widehat{\Gamma}^A_{\sigma\mu\nu}$ is totally antisymmetric and proportional to an axial vector A_μ,

$$\widehat{\Gamma}^A_{\sigma\mu\nu} = \varepsilon_{\sigma\mu\nu\rho}\ A^\rho \tag{5.10}$$

and $\widehat{\Gamma}^T_{\sigma\mu\nu}$ is traceless. For our subsequent purposes – the discussion of cosmologies – it suffices to confine ourselves to torsion of the type $\widehat{\Gamma}^V_{\sigma\mu\nu}$ and $\widehat{\Gamma}^A_{\sigma\mu\nu}$ [22]. Moreover, according to the symmetries associated to the cosmological principle (isotropy and homogeneity), only the time (zero) components of V_μ and A_μ are assumed to be nonvanishing and to depend on t only.

First, we make an ansatz for the metric g analogous to eq. (5.1),

$$ds^2 = -dt^2 + a^2(t)\ d\vec{x}^2\ . \tag{5.11}$$

Then it turns out to be convenient to parametrize the nonvanishing components of $\widehat{\Gamma}^V_{\sigma\mu\nu}$ as

$$\widehat{\Gamma}^V_{0ij} = -\widehat{\Gamma}^V_{i0j} = \delta_{ij}\ a^2(t)\ \gamma(t)\ , \tag{5.12}$$

and the nonvanishing components of $\widehat{\Gamma}^A_{\sigma\mu\nu}$ as

$$\widehat{\Gamma}^A_{ijk} = \varepsilon_{ijk}\ a^3(t)\ \beta(t)\ . \tag{5.13}$$

Assuming the existence of an action S, the full connection $\Gamma^\sigma{}_{\mu\nu}$ and the metric $g_{\mu\nu}$ are determined by varying

$$S = \int d^4x \left\{ \frac{1}{2}\sqrt{-g}\ g^{\mu\nu}\ R^\sigma{}_{\mu\nu\sigma}(\Gamma) + \mathcal{L}_m(g,\Gamma,\cdots) \right\} \tag{5.14}$$

both with respect to $g_{\mu\nu}$ and $\Gamma^\sigma{}_{\mu\nu}$ [21, 22] (where $\mathcal{L}_m$ is the Lagrangian of matter fields). In the context of cosmology suitable averages over the matter fields are performed, and the resulting equations can be expressed in terms of an "effective" density (depending on the

torsion), an "effective" pressure and sources for torsion, whose unknown properties allow to treat the functions $\gamma(t)$, $\beta(t)$ in eqs. (5.12) and (5.13) as additional arbitrary parameters [22, 23].

Here we are not interested in the dynamics that fixes $a(t)$, $\gamma(t)$ and $\beta(t)$, but rather in the following problem: Given the three above functions, we can construct the Riemann tensor (5.5) or its version R_{abcd} according to (2.1). Then we can find its dual according to eq. (3.3a) and ask, whether the components of $\widetilde{R}_{abcd}$ can be of the form of eqs. (5.3) such that they describe standard – torsionless – FRW cosmologies.

This is a highly nontrivial question, since $R^{\sigma}{}_{\mu\nu\rho}$ has none of the properties (2.2) – (2.4) due to the presence of torsion. (Of course, we introduced torsion in order to avoid the cyclic identity (2.3) which implies the vanishing of $\widetilde{R}^{a}{}_{b}$, but now it can well be impossible to satisfy all of the constraints (2.2), (2.4) for $\widetilde{R}_{abcd}$.)

First, $R^{\sigma}{}_{\mu\nu\rho}$ as obtained from eq. (5.5) with $\Gamma^{\sigma}{}_{\mu\nu}$ as in eq. (5.6), a metric as in (5.11) and $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$ given by the sum of eqs. (5.12) and (5.13), is no longer symmetric:

$$R_{\sigma\mu\nu\rho} \neq R_{\nu\rho\sigma\mu} \text{ (in general)} . \tag{5.15}$$

Consequently the result for $\widetilde{R}_{abcd}$ depends on whether, in the duality transformation, one contracts ε_{abcd} to the left of R_{abcd} (as in eq. (3.3a)), to the right of R_{abcd}, or whether one employs a left-right symmetric definition of the duality transformation. Below we will treat all possible cases.

Recall that, originally, the duality transformation (3.3a) leads to flat Minkowski space (described by $\widetilde{R}_{abcd}$) if the field strength F_{abcd} vanishes, regardless of the cosmological constant (curvature of (A)dS space) described by R_{abcd}. We will continue to work with the assumption of vanishing F_{abcd}. However, in order to treat the different possible duality transformations simultaneously, we generalize (3.3a) as

$$\widetilde{R}_{abcd} = \frac{1}{4}\left[(1+e)\varepsilon_{abef}\ R^{ef}{}_{cd} + (1-e)R_{ab}{}^{ef}\ \varepsilon_{efcd}\right] + \frac{1}{12}\ \varepsilon_{abcd}\ R\ . \tag{5.16}$$

We have dropped the terms $\sim F_{abcd}$, but the parameter e allows to interpolate between

i) left duality ($e = 1$)

ii) right duality ($e = -1$)

iii) left-right symmetric duality ($e = 0$).

Our results concerning the properties of $\widetilde{R}_{abcd}$ are as follows: First, $\widetilde{R}_{abcd}$ satisfies all of the symmetry properties (2.2) (where the last one is nontrivial) if and only if the three functions α, β and γ satisfy

$$e\left(\beta^2 - \gamma^2 + \dot{\gamma} - \dot{\alpha}\gamma + \ddot{\alpha}\right) = 0 \ . \tag{5.17}$$

Second, $\widetilde{R}_{abcd}$ satisfies the cyclic identity (2.3) if and only if

$$e\left(\beta^2 - \gamma^2 + \dot{\gamma} - \dot{\alpha}\gamma + \ddot{\alpha}\right) = 0 \ . \tag{5.18}$$

The fact that eqs. (5.17) and (5.18) coincide is not trivial; the presence of the last term $\sim \varepsilon_{abcd} R$ in (5.16) is crucial to this end. Then it is quite remarkable that a very large number of constraints is satisfied simultaneously once either $e = 0$, or one particular relation between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ is satisfied.

In terms of the *original* Ricci tensor $R^a{}_b$ (before the duality transformation) the relation (5.17) corresponds to $R_{ii} = -R_{00}$ (no sum over i), or

$$R_{ab} = \lambda(t)\ \eta_{ab} \tag{5.19}$$

for some function

$$\lambda(t) = -3\left(\dot{\alpha}^2 + \ddot{\alpha} + \dot{\alpha}\gamma + \dot{\gamma}\right) \ . \tag{5.20}$$

Once eq. (5.17) holds, the only nonvanishing components of $\widetilde{R}_{abcd}$ are

$$\widetilde{R}_{ijij} = \frac{1}{2}\ (1+e)\dot{\beta} + (1-e)\beta\gamma + \frac{(3-e)}{2}\dot{\alpha}\beta \ , \tag{5.21a}$$

$$\widetilde{R}_{i0i0} = -\frac{1}{2}\ (1-e)\dot{\beta} - (1+e)\beta\gamma - \frac{(3-e)}{2}\dot{\alpha}\beta \ . \tag{5.21b}$$

Instead of investigating the validity of the second Bianchi identity (2.4) for $\widetilde{R}_{abcd}$, we can study directly whether eqs. (5.21) can coincide with eqs. (5.3), that would describe a FRW cosmology in terms of $\widetilde{R}_{abcd}$.

First, we find that for $e = 0$ (left-right symmetric duality) the two expressions (5.21a) and (5.21b) coincide up to a sign, which implies, from eqs. (5.3), that $\ddot{\widetilde{\alpha}} = 0$ or

$$\dot{\widetilde{\alpha}} = \text{ const. } = \pm H \ . \tag{5.22}$$

Hence, the left-right symmetric dual of a cosmology with torsion corresponds necessarily to (A)dS, what is not general enough for our purposes.

On the other hand, for both cases $e = \pm 1$ we can describe *any* cosmology $\widetilde{\alpha}(t)$ if, in addition to eq. (5.17), the three functions α, β and γ satisfy the following relation:

From eqs. (5.3) one can derive

$$\widetilde{R}_{ijij} + \widetilde{R}_{i0i0} = \frac{d}{dt}\left(\widetilde{R}_{ijij}\right)^{1/2} \ , \tag{5.23}$$

and – after the use of eqs. (5.21) – the satisfaction of the corresponding additional differential equation between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ is sufficient in order to be able to write eqs. (5.21) in the form of eqs. (5.3) with $\dot{\widetilde{\alpha}} = \left(\widetilde{R}_{ijij}\right)^{1/2}$ and $\left(\widetilde{R}_{ijij}\right)^{1/2}$ as in eq. (5.21a). Since, for $\widetilde{\alpha}(t)$ given, we have only two equations (5.17) and (5.23) to satisfy, the remaining freedom allows to describe any cosmology $\widetilde{\alpha}(t)$ with the help of suitable functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$.

Generally, an explicit solution of the corresponding system of differential equations (with $\widetilde{\alpha}(t)$ given) is very difficult to impossible. However, FRW cosmologies corresponding to a relativistic fluid with a simple equation of state, $p = w\rho$, give rise to logarithmic scale factors $\widetilde{\alpha}(t)$ with

$$\dot{\widetilde{\alpha}}(t) = \frac{2}{3}\,(1+w)t^{-1} \ . \tag{5.24}$$

In this case all required relations can be satisfied by a simple ansatz

$$\dot{\alpha}(t) = a_0\ t^{-1}$$

$$\beta(t) = b_0 \ t^{-1}$$
$$\gamma(t) = g_0 \ t^{-1} \tag{5.25}$$

and the (e-dependent) solution of a non-linear algebraic system of 3 equations for the 3 constants a_0, b_0 and g_0.

The main result of the present section is, however, the statement made already above: Given a duality transformation of the form of eq. (5.16), with $e = \pm 1$, we can obtain any FRW-like Riemann tensor $\widetilde{R}_{abcd}$ as the dual of an "original" theory with torsion of the form in eqs. (5.12) and (5.13), for suitable functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$. The fact that we manage to satisfy all symmetry conditions and Bianchi identities for $\widetilde{R}_{abcd}$ simultaneously is highly nontrivial, and depends on the last term in eq. (3.3a) which can be considered as a remnant of the duality transformation including the 3-form field (although its field strength has finally been set to zero).

Except for the relation (5.19) we have not been able to express the required relations between $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ in terms of dynamical principles of the original theory with torsion. If these relations do not hold (exactly), the resulting dual Riemann tensor $\widetilde{R}_{abcd}$ corresponds again to a cosmology with torsion, a possibility that has been investigated, e.g., in [22, 23].

6 Discussion

Above we considered generalizations of gravitational S-duality. Including a 3-form field A_{abc}, we obtained the particularly interesting result that the dual cosmological constant vanishes independently of the value of the "original" cosmological constant, if the corresponding field strength vanishes. This motivated us to investigate under which conditions phenomenologically relevant metrics $\widetilde{g}_{\mu\nu}$ can be obtained through gravitational S-duality transformations.

We found that the Schwarzschild metric can be obtained as the dual of a contracted Taub-NUT-AdS metric. The necessity to perform such a contraction can be considered as unsatisfactory, but otherwise one is left with a non-vanishing NUT parameter $\widetilde{\ell}$ in the dual (supposedly physical) metric. The physics and phenomenology of non-vanishing NUT

parameters has been studied in [24]. However, non-vanishing NUT parameters give rise to closed timelike curves, if one insists on the completeness of the metric [20], which seems to rule out such a possibility. But, for tiny NUT parameters ℓ, the argument assumes completeness of the metric at tiny (timelike) distances. Assuming a modification of gravity (UV regularization) at small distances, this problem may disappear. For instance, lattice gauge theories contain Dirac monopoles (whose Dirac string "escapes" through the space between the lattice sites). It seems to be a logical possibility that lattice regularized theories of gravity contain equally configurations with "magnetic" masses (corresponding to NUT parameters ℓ), without the above problem of closed timelike curves. Then we could possibly live with small nonvanishing NUT charges ℓ (up to phenomenological constraints [24]), and the contraction performed at the end of chapter 4 does not have to be pushed to its singular limit.

In chapter 5 we obtained FRW-like metrics as duals of theories with torsion. Clearly, the corresponding Riemann tensor (5.5), given the decomposition (5.6) of the connection, can always be written as

$$R^{\sigma}{}_{\mu\nu\rho} = {}^{M}R^{\sigma}{}_{\mu\nu\rho} + \widehat{R}^{\sigma}{}_{\mu\nu\rho} \tag{6.1}$$

where ${}^{M}R^{\sigma}{}_{\mu\nu\rho}$ depends on the metric $g_{\mu\nu}$ only, and $\widehat{R}^{\sigma}{}_{\mu\nu\rho}$ depends on the contribution of torsion to the connection. Thus the presence of torsion in $R^{\sigma}{}_{\mu\nu\rho}$ – appearing on the right-hand side of the duality transformation (5.16) – can equally be interpreted as another generalization of the original duality transformation rule (3.4a) in the form of adding more matter (torsion) dependent terms to its right-hand side. However, here matter is not represented in the form of fields, but in the form of components of the torsion $\widehat{\Gamma}^{\sigma}{}_{\mu\nu}$, that are treated as effective densities as it is the case for ρ and the pressure p in FRW cosmologies with matter in the form of a relativistic liquid.

Clearly, an application of the present results on metrics (that are related by duality) to the cosmological constant problem relies on the possibility to extend the duality transformations in the gravitational sector consistently to a complete theory including matter: The result $\widetilde{\Lambda} = 3\Sigma$ in section 3 has to be consistent with the properties of the dual stress energy tensor $\widetilde{T}_{ab}$. We remark, however, that the particular case $\widetilde{T}_{ab} = 0$ (in the vacuum) could correspond

to a situation where the coupling of matter to the dual metric is only describable in terms of point like matter, but not (in a local way) in terms of matter in the form of fields. Such a situation would be similar to the coupling of magnetic monopoles to an abelian gauge field A, or to the coupling of electric monopoles to its dual $\widetilde{A}$.

Admittedly such a possibility seems far-fetched, but we recall that tests of general relativity (or its Newtonian limit) involve only the coupling of point like matter to gravity, down to length scales of ~ 1 mm. (A notable exception is the electromagnetic field, which has to couple to the dual metric $\widetilde{g}$ within this scenario, such that light propagates along geodesics corresponding to $\widetilde{g}$.)

In the absence of a duality transformation including matter (that may well be non local, and may even require to review our present concepts of space time) we confined ourselves to a "bottom-up" approach, in the sense that we studied macroscopic configurations of the metric (verified at distances $\gtrsim 1$ mm), where matter is represented either as a point like source at the center of the Schwarzschild solution, or as effective densities in the case of cosmological solutions.

The fact that both phenomenologically relevant metrics can be obtained, under suitable assumptions, as S-duals indicates that our observed space time could possibly be identified with the dual of some underlying metric.

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